

The asymptotic density of dead ends in non-amenable groups

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Abstract

We show that, in non-amenable groups, the density of elements of depth at least d goes to 0 exponentially in d .

Let G be a group, A a finite generating set for G . Then the *depth* (more verbosely the *dead-end depth*) of $g \in G$ is the distance (in the word metric with respect to A) from g to the complement of the radius- $d_A(1, g)$ closed ball about the origin. While many common examples of (group, generating set) pairs admit a uniform bound on the depth of elements, some do not; we say these groups have *deep pockets*. The standard example of a group with deep pockets is the lamplighter group with respect to the standard generating set; see [1].

Although, by definition, groups with deep pockets have elements of arbitrarily large depth, this leaves open the question of how many there are. In a remark at p. 91 of [2], Kyoji Saito asked (in the context of the study of so-called pre-partition functions) under what circumstances the asymptotic density of elements of depth > 1 is guaranteed to be 0; he had posed the same question in a personal communication to the author in 2009. Inspired by this question (but as yet unable to answer it), we show that the density of elements of depth at least m approaches 0 exponentially in m , provided the group is not amenable.

We begin with a lemma from analysis.

Lemma 1. *Let $f: (0, 1) \rightarrow (0, \infty]$ be completely arbitrary. Then there exists U an open interval in $(0, 1)$ and $\epsilon > 0$ such that, for every $x_0 \in U$ and $\delta > 0$, there is $x \in (0, 1)$ with $|x - x_0| < \delta$ and $f(x) > \epsilon$.*

Proof. For every $\epsilon > 0$, let U_ϵ be the set of x_0 such that, for all $\delta > 0$, there exists an appropriate x . We will show that some U_ϵ contains an open interval.

We know that $x_0 \in U_{f(x_0)/2}$ for every x_0 (just set $x = x_0$), so $\bigcup_{\epsilon} U_{\epsilon} = (0, 1)$, since the range of f does not include 0. But, if $\epsilon_1 < \epsilon_2$, then $U_{\epsilon_1} \supseteq U_{\epsilon_2}$, so

$$\bigcup_{\epsilon} U_{\epsilon} = \bigcup_{\epsilon \in \mathbb{Q}} U_{\epsilon}.$$

This union is countable, so, since it exhausts $(0, 1)$, Baire's Theorem says that some U_{ϵ} is dense on some interval. But, for every ϵ , U_{ϵ} is closed in $(0, 1)$. Thus, if it is dense on an (open) interval, it includes that interval, so we are done. \square

Corollary 2. *Let $f: (0, 1) \rightarrow (0, \infty]$ be completely arbitrary. Then there is $A > 1$ such that for every sufficiently large integer m there exists $S \subset (0, 1)$ such that any two elements of S differ from each other, and from 0 and 1, by at least $1/m$ and $\prod_{x \in S} [1 + f(x)] > A^m$.*

Proof. Choose U and $\epsilon > 0$ as in Lemma 1. Let L be the length of U . If $m \in \mathbb{N}$, then U contains $\lfloor Lm \rfloor$ disjoint subintervals of length $1/m$. Let S consist of one point each from the even-numbered subintervals other than the last (that is from every other subinterval, omitting the first and last), chosen such that $f(x) > \epsilon$ for all $x \in S$; this is possible by construction of U . Then clearly any two elements of S differ from each other, and from 0 and 1, by at least $1/m$. But

$$\prod_{x \in S} [1 + f(x)] \geq (1 + \epsilon)^{\lfloor Lm/2 \rfloor - 1} > (1 + \epsilon)^{Lm/2 - 2} > A^m$$

for m sufficiently large, so long as $A < (1 + \epsilon)^{L/2}$. \square

Lemma 3. *Let G be a group, A a finite generating set for G and $m, n \in \mathbb{N}$. Let $h \in S_n$. Then h has depth at least m iff h is at distance at least m from B'_n (where by B'_n we mean the complement of B_n in G).*

This is just the definition of depth.

Proposition 4. *Let G be a non-amenable group and A a finite generating set for G . Let B_n (respectively S_n) be the (closed) ball (resp. sphere) of radius n about the identity in G with respect to A . Let D_m be the set of elements of G of depth at least m with respect to A . Then there are $a > 0$ and $0 < b < 1$ such that, for all m , $\sup_{n > m} |S_n \cap D_m| / |B_n| < ab^m$.*

Proof. It follows from Lemma 3 that, for any m ,

$$\sup_{n > m} \frac{|S_n \cap D_m|}{|B_n|}$$

is the supremum over all $n > m \in \mathbb{N}$ of the fraction of elements of B_n that are in S_n and at distance at least m from B'_n . Thus there exists a sequence $n_m > m \in \mathbb{N}$ such that the fraction (say f_m) of elements of B_{n_m} that are in S_{n_m} and at distance at least m from B'_{n_m} is at least $\sup_{n > m} |S_n \cap D_m| / |B_n| - \epsilon/2^m$.

I claim that $f_m \leq ab^m$ for some $a > 0$ and $0 < b < 1$ independent of m ; this will imply the theorem (after increasing a slightly and possibly b). For each

$\delta \in (0, 1)$, let $H_{m,\delta} \subset B_{n_m}$ be the set of points at distance $> 1 + \delta(m-1)$ from B'_{n_m} . Every point of $\partial H_{m,\delta}$ is in B_{n_m} and at distance either $1 + \lfloor \delta(m-1) \rfloor$ or $2 + \lfloor \delta(m-1) \rfloor$ from B'_{n_m} . For each δ , we know that, since neither $H_{m,\delta}$ nor any subsequence thereof is Følner,

$$\liminf_{m \rightarrow \infty} \frac{|\partial H_{m,\delta}|}{|H_{m,\delta}|} > 0,$$

whence

$$\inf_m \frac{|\partial H_{m,\delta}|}{|H_{m,\delta}|} = l(\delta) > 0,$$

since $n_m > m$ guarantees that $H_{m,\delta}$, hence $\partial H_{m,\delta}$, is never empty. Note that $l(\delta)$ is independent of m .

It follows by Corollary 2 that there is $A > 1$ (also independent of m) such that for all sufficiently large $N \in \mathbb{N}$ there exists $C_N \subset (0, 1)$ such that any two elements of C_N differ from each other, and from 0 and 1, by at most $1/N$ and $\prod_{\delta \in C_N} [1 + l(\delta)] > A^N$. We thus have

- $\prod_{\delta \in C_N} [1 + l(\delta)] > A^N$
- for all $m > 2N + 1$, $\{\partial H_{m,\delta} \mid \delta \in C_N\}$ are pairwise disjoint subsets of $B_{n_m} - \bigcap_{\delta \in (0,1)} H_{m,\delta}$ and
- for all $m \in \mathbb{N}$, $\delta \in (0, 1)$,

$$\frac{|\partial H_{m,\delta}|}{|H_{m,\delta}|} \geq l(\delta).$$

It follows that, for all $m > 2N + 1$, the fraction of elements of B_{n_m} belonging to

$$S_{n_m} \cap D_m \subset \bigcap_{\delta \in (0,1)} H_{m,\delta}$$

is at most

$$\prod_{\delta \in C_N} \frac{|H_{m,\delta} - \partial H_{m,\delta}|}{|H_{m,\delta} \cup \partial H_{m,\delta}|} < \prod_{\delta \in C_N} \frac{|H_{m,\delta}|}{|H_{m,\delta}| + |\partial H_{m,\delta}|} \leq \prod_{\delta \in C_N} \frac{1}{1 + l(\delta)} < A^{-N}.$$

Since N was arbitrary, we have that, for all m , $f_m < A^{(1-m)/2}$, so the claim is proven and we are done. \square

Theorem 5. *Let G be a non-amenable group and A a finite generating set for G . Let B_n be the (closed) ball of radius n about the identity in G with respect to A . Let D_m be the set of points of depth at least m with respect to A . Then there are $c > 0$ and $0 < b < 1$ such that, for all m ,*

$$\lim_{n \rightarrow \infty} \frac{|B_n \cap D_m|}{|B_n|} < cb^m.$$

Proof. Since neither the B_i nor any subsequence thereof are Følner, we have

$$\frac{|B_{i-2}|}{|B_i|} < L < 1$$

for all i , where L depends only on G and A . Thus $|B_{i-j}|/|B_i| < L^{\lfloor j/2 \rfloor} < L^{j/2-1}$, since certainly $|B_i| \leq |B_{i+1}|$ for all i . Thus, for any $m < n$,

$$\begin{aligned} \frac{|B_n \cap D_m|}{|B_n|} &= \sum_{i=0}^n \frac{|S_i \cap D_m|}{|B_n|} = \sum_{i=0}^n \frac{|S_i \cap D_m|}{|B_i|} \frac{|B_i|}{|B_n|} \\ &= \frac{|B_m|}{|B_n|} + \sum_{i=m+1}^n \frac{|S_i \cap D_m|}{|B_i|} \frac{|B_i|}{|B_n|} < \frac{|B_m|}{|B_n|} + \sum_{i=m+1}^n ab^m \frac{|B_i|}{|B_n|} \\ &< \frac{|B_m|}{|B_n|} + \sum_{i=m+1}^n ab^m L^{(n-i)/2-1} < \frac{|B_m|}{|B_n|} + ab^m \sum_{i=0}^{\infty} L^{i/2-1} = \frac{|B_m|}{|B_n|} + cb^m/2, \end{aligned}$$

where the first inequality is by Proposition 4 and the second by the preceding sentence. (Here $c = 2a \sum_{i=0}^{\infty} L^{i/2-1}$ depends only on G and A .) Thus

$$\limsup_{n \rightarrow \infty} \frac{|B_n \cap D_m|}{|B_n|} \leq cb^m/2 < cb^m$$

for all m , as claimed. □

References

- [1] S. Cleary and J. Taback. Dead end words in lamplighter groups and other wreath products. *The Quarterly Journal of Mathematics*, 56(2):165–78, 2005. [arXiv:math.GR/0309344](#).
- [2] K. Saito. Limit elements in the configuration algebra of a cancellative monoid. *Publications of the Research Institute for Mathematical Sciences*, 46:37–113, 2010.